MATHEMATICS

ORDER CONTINUOUS MAPPINGS ON KÖTHE SPACES. III

BY

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1. *Introduction*

It is assumed that the reader is familiar with the results contained in MULLINS [5]. In the last section of that paper we began an investigation of the various relationships between certain classes of linear mappings from one Köthe space into another. In particular we considered the class of order continuous linear mappings, the class of order bounded linear mappings and the class of weakly continuous linear mappings. In the present paper we continue these investigations and consider also the class $L^N(X_1, X_2)$ of linear mappings that are continuous for the topology of uniform convergence on order bounded sets (see PERESSINI [6], section I). We have found that our results hold for a larger class of spaces than Köthe spaces. For this reason we use the more general setting of partially ordered vector spaces. Our main result (Theorem 2.8) is that for a certain class of spaces, which includes the Köthe spaces, the class of (order) bounded order continuous linear mappings, the class of (order) bounded weakly continuous linear mappings and $L^N(X_1, X_2)$ all coincide.

2. *Linear Mappings on Partially Ordered Spaces*

Let X be a partially ordered vector space over the real numbers, and let K denote the positive cone of X. We shall use $X^\wedge$ to denote the vector space of all linear functionals on X, defining addition and scalar multiplication in the usual way. The *positive cone* $K^\wedge$ in $X^\wedge$ consists of those linear functionals ϕ for which $\phi(x) \geqslant 0$ whenever $x \in K$. The linear hull $K^\wedge - K^\wedge$ of $K^\wedge$ is denoted by X^+. A linear functional on X is *order bounded* if it maps an order bounded set in X onto a bounded set of real numbers. X^b denotes the subspace of $X^\wedge$ consisting of all order bounded linear functionals on X. Note that X^+ is always a subset of X^b. If X is a vector lattice, then $X^+ = X^b$ and X^b is an order complete vector lattice for the partial ordering determined by the positive cone $K^\wedge$. It is possible for X^b to contain only the zero functional even for rather simple vector lattices. *However, throughout this paper we will insist that X^b separates the points of X.* This requirement ensures that the space X and X^b are placed in duality by the mapping

$$\langle x, \phi \rangle = \phi(x)$$

(we shall use $\langle \cdot, \cdot \rangle$ to denote both the dual system and the bilinear form associated with the dual system).

Another useful idea associated with a vector lattice X is that of a *solid* set, that is, a subset S of X such that $u \in S$ and $|x| \leqslant |u|$, $x \in X$, imply that $x \in S$. If a solid set S is also a subspace of X, then it is called an *ideal.* Whenever X is order complete, an ideal S is a *band* if $A \subset S$ and $\sup A = x \in X$ imply that $x \in S$.

Now, let Y be a vector space which is in duality with the space X and such that the set $K - K$ is $\sigma(X, Y)$—dense in X. Then the *dual cone* K' in Y, which is defined by

$$K' = \{y \in Y : \langle x, y \rangle \geqslant 0 \text{ for all } x \in K\},$$

determines a partial ordering on Y. Let $\mathfrak{S}_0$ denote the collection of all subsets of Y which are bounded with respect to this partial ordering. The *order bound topology* on X is the topology of uniform convergence on the elements of $\mathfrak{S}_0$. From the general theory of $\mathfrak{S}$-topologies it follows that the order bound topology on X is compatible with the linear structure of X if and only if each element of X, considered as a linear functional on Y, is order bounded (see Bourbaki [1], Chap. III, § 3, Prop. 1). When this condition is satisfied, we shall denote the $\mathfrak{S}_0$-topology on X by $o(X, Y)$. In particular if the cone K is generating in X, that is, if $X = K - K$, then $X \subset Y^b$. Also, whenever $Y = K' - K'$, the polars of the elements of $\mathfrak{S}_0$ form a θ-neighborhood basis for the order bound topology and the class $\{[-u, u]\}_{u \in K'}$ is a fundamental system for $\mathfrak{S}_0$.

Henceforth, $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ will denote dual systems over the real numbers. X_1 and X_2 will be vector lattices ordered by positive cones K_1 and K_2, respectively. We will assume that the dual cones K_1' and K_2' are generating in their respective spaces Y_1 and Y_2. We are interested in the following classes of linear mappings from X_1 into X_2:

(i) The set $L^b(X_1, X_2)$ of all *order bounded linear mappings*, that is, those linear mappings that map order bounded sets in X_1 onto order bounded sets in X_2.

(ii) The set $L^+(X_1, X_2)$ of all linear mappings that can be written as the difference of two positive linear mappings from X_1 into X_2 (a mapping T from X_1 into X_2 is *positive* if $T(K_1) \subset K_2$). We remark that $L^b(X_1, X_2) = L^+(X_1, X_2)$ whenever X_2 is order complete.

(iii) The set $L^O(X_1, X_2)$ (resp., $L^{SO}(X_1, X_2)$) of all *order continuous* (resp., *order sequentially continuous*) *linear mappings*, that is, those linear mappings that map nets (resp., sequences) in X_1 which order converge to zero onto nets (resp., sequences) in X_2 which also order converge to zero.

(iv) The set $L(X_1, X_2)$ of all weakly continuous linear mappings.

(v) The set $L^N(X_1, X_2)$ of all linear mappings that are continuous for the order bound topologies $o(X_1, Y_1)$ and $o(X_2, Y_2)$.

If T is a linear mapping from X_1 into X_2, then the *adjoint* T' of T is defined by

$$\langle x, T'u\rangle = \langle Tx, u\rangle$$

where $x \in X_1$ and $u \in Y_2$. It should be pointed out that T' is a mapping from Y_2 into $X_1^{\wedge}$, and if $T \in L^b(X_1, X_2)$ then T' maps Y_2 into X_1^b since $Y_2 \subset X_2^b$. The range of T' is a subset of Y_1 if and only if T is weakly continuous.

We are now ready to give the first series of results which establishes some relationships between order bounded linear mappings, weakly continuous linear mappings and linear mappings that are continuous for the order bound topologies.

Proposition 2.1. *If Y_1 is an ideal in X_1^b and T is a linear mapping from X_1 into X_2, then*

$$T \in L^N(X_1, X_2) \Leftrightarrow T' \in L^b(Y_2, Y_1).$$

Proof. This result follows immediately from Theorem 4.1 in PERESSINI [6]. It is only necessary to point out that T is weakly continuous whenever $T \in L^N(X_1, X_2)$ since, by Theorem 1.2 in [6], the topology $o(X_1, Y_1)$ is consistent with the dual system $\langle X_1, Y_1\rangle$. Q.E.D.

Proposition 2.2. *If X_2 is order complete, then*

$$T \in L^b(X_1, X_2) \Rightarrow T' \in L^b(Y_2, X_1^b).$$

Proof. Let $T \in L^b(X_1, X_2)$. Then T can be written in the form $T = T_1 - T_2$, where T_1 and T_2 are positive (and hence order bounded) linear mappings from X_1 into X_2. Thus $T' = T_1' - T_2'$, where T_1' and T_2' are positive linear mappings from Y_2 into X_1^b, so that $T' \in L^+(Y_2, X_1^b) = L^b(Y_2, X_1^b)$. Q.E.D.

Proposition 2.3. *If X_2 is order complete and Y_1 is a band in X_1^b, then*

$$T \in L(X_1, X_2) \cap L^b(X_1, X_2) \Rightarrow T' \in L^b(Y_2, Y_1).$$

Proof. Let $T \in L(X_1, X_2) \cap L^b(X_1, X_2)$. Then $T' \in L^b(Y_2, X_1^b)$ so that for any bounded set B in Y_2, the set $\{T'u : u \in B\}$ is bounded in X_1^b. Since T is weakly continuous, we have that $T'(Y_2) \subset Y_1$. But Y_1 is a band in $X_1{}^b$, so that the set $\{T'u: u \in B\}$ must also be bounded in Y_1. Therefore $T' \in L^b(Y_2, Y_1)$. Q.E.D.

Proposition 2.4. *If X_2 is a band in $Y_2{}^b$, Y_2 is a sublattice of $X_2{}^b$, and Y_1 is an ideal in $X_1{}^b$, then*

$$T \in L^N(X_1, X_2) \Rightarrow T \in L^b(X_1, X_2).$$

Proof. Let $T \in L^N(X_1, X_2)$. By Proposition 2.1 we obtain $T' \in L^b(Y_2, Y_1)$. Since $T = (T')'$ it follows from Proposition 2.3 that $T \in L^b(X_1, X_2)$. Q.E.D.

Proposition 2.5. *If X_2 is a band in $Y_2{}^b$, Y_2 is a sublattice of $X_2{}^b$, and Y_1 is a band in $X_1{}^b$, then*

$$T \in L^N(X_1, X_2) \Leftrightarrow T \in L(X_1, X_2) \cap L^b(X_1, X_2).$$

Proof. Propositions 2.1, 2.3 and 2.4. Q.E.D.

Having found conditions which ensure that $L^N(X_1, X_2) = L(X_1, X_2) \cap \cap L^b(X_1, X_2)$, we turn our attention to the relationship between $L^N(X_1, X_2)$ and $L^O(X_1, X_2)$. In particular, we will give conditions which imply that $L^N(X_1, X_2) = L^O(X_1, X_2) \cap L^b(X_1, X_2)$. In order to obtain this result we will make use of some properties of X^O, the subspace of $X^\wedge$ consisting of all order continuous linear functionals on the vector lattice X. Hence we will first mention those properties of X^O which will be needed. A thorough examination of these ideas can be found in Vulikh ([7], Chapters VIII and IX).

If X^O is equipped with the partial ordering induced by $K^\wedge$, then it becomes a band in X^b (this result can also be found in Luxemburg and Zaanen [4], Note VIII, Theorem 27.2). The elements of X determine linear functionals on X^b in the obvious way and these linear functionals are order continuous, i.e. $X \subset (X^b)^O$. Since X^O is a band in X^b, we may also write $X \subset (X^O)^O$. Whenever $X = (X^O)^O$, we say that X is *perfect*. If X^O contains an ideal Y which separates the points of X, then it can be shown that a point $x \in X$ is positive if and only if $\phi(x) \geqslant 0$ for all positive $\phi \in Y$ (see Vulikh [7], page 287). From this result it follows that for such a Y the positive cone K of X is weakly closed with respect to the dual system $\langle X, Y \rangle$. With these ideas in mind we proceed with our study of linear mappings by considering $L^O(X_1, X_2)$.

Proposition 2.6. *If $X_1{}^O \subset Y_1$ and $Y_2 \subset X_2{}^O$ then*

$$T \in L^O(X_1, X_2) \Rightarrow T \in L(X_1, X_2).$$

Proof. Let $T \in L^O(X_1, X_2)$. If $(x_\alpha)_{\alpha \in \mathscr{A}}$ is a net in X_1 which order converges to θ, then $(Tx_\alpha)_{\alpha \in \mathscr{A}}$ order converges to θ in X_2. Since $Y_2 \subset X_2{}^O$, for any $w \in Y_2$ we have

$$\lim \langle x_\alpha, T'w \rangle = \lim \langle Tx_\alpha, w \rangle = 0.$$

Hence, $T'w \in X_1{}^O \subset Y_1$, so that $T'(Y_2) \subset Y_1$. Therefore, $T \in L(X_1, X_2)$. Q.E.D.

Recall that if X_2 is an order complete vector lattice, then every element of $L^b(X_1, X_2)$ can be written as the difference of two positive linear mappings from X_1 into X_2. In fact, $T = T^+ - T^-$ where T^+ is defined for $x \in K_1$ by

$$T^+x = \sup \{Tz : z \in X_1 \text{ and } \theta \leqslant z \leqslant x\}$$

and then extended to all of X_1 by using $X_1 = K_1 - K_1$; and, T^- is simply $(-T)^+$. The linear mappings T^+ and T^- are called, respectively, the *positive* and *negative parts* of T. The *absolute value* $|T|$ of T is given by $|T| = T^+ + T^-$.

Proposition 2.7. *Let X_2 be an order complete vector lattice for which the positive cone K_2 is $\sigma(X_2, Y_2)$-closed, and let $Y_1 \subset X_1{}^O$. If $T \in L^b(X_1, X_2) \cap$ $\cap\, L(X_1, X_2)$, then T^+, T^-, T and $|T|$ are all elements of $L^O(X_1, X_2)$.*

Proof. Let $T \in L^b(X_1, X_2) \cap L(X_1, X_2)$. As remarked above, for $x \in K_1$

$$T^+x = \sup \{Tz : z \in X_1 \text{ and } \theta \leqslant z \leqslant x\}.$$

We will show first that whenever $(x_\alpha)_{\alpha \in \mathscr{A}}$ is a net in K_1 which increases to $x_0 \in K_1$, then $(T^+x_\alpha)_{\alpha \in \mathscr{A}}$ increases to T^+x_0 in X_2. We set

$$y_0 = \sup_{\alpha \in \mathscr{A}} T^+x_\alpha.$$

Since T^+ is a positive mapping, we know that $y_0 \leqslant T^+x_0$. Now, let z be any element of X_1 such that $\theta \leqslant z \leqslant x_0$. We define a net $(z_\alpha)_{\alpha \in \mathscr{A}}$ which increases to z in X_1 by setting $z_\alpha = \inf \{x_\alpha, z\}$ for each $\alpha \in \mathscr{A}$. Since $Y_1 \subset X_1{}^O$, it follows that $(z_\alpha)_{\alpha \in \mathscr{A}}$ converges weakly to z in X_1 so that $(Tz_\alpha)_{\alpha \in \mathscr{A}}$ converges weakly to Tz in X_2. But

$$Tz_\alpha \leqslant T^+x_\alpha \leqslant y_0$$

for all $\alpha \in \mathscr{A}$, so the weak closedness of the positive cone K_2 implies that $Tz \leqslant y_0$. Since z was an arbitrary element of X_1 satisfying the inequality $\theta \leqslant z \leqslant x_0$, we obtain $T^+x_0 \leqslant y$. It is an immediate consequence of the linearity of T^+ that $T^+ \in L^O(X_1, X_2)$. Since $T^- = (-T)^+$ it also follows that $T^- \in L^O(X_1, X_2)$. Therefore $T = T^+ - T^-$ and $|T| = T^+ + T^-$ are also elements of $L^O(X_1, X_2)$. Q.E.D.

The dual systems $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ determine weak topologies, not only on the spaces X_1 and X_2, but also on Y_1 and Y_2. Hence, we may consider the class $L(Y_2, Y_1)$ of all linear mappings from Y_2 into Y_1 that are continuous for the weak topologies $\sigma(Y_2, X_2)$ and $\sigma(Y_1, X_1)$. Furthermore, if the cones K_1' and K_2' make Y_1 and Y_2, respectively, into vector lattices, then we can also consider the class $L^O(Y_2, Y_1)$ of order continuous linear mappings from Y_2 into Y_1. These remarks lead us to the following corollary to Proposition 2.7.

Corollary 2.7. *Let Y_1 and Y_2 be ideals in X_1^b and X_2^b, respectively. If $S \in L(Y_2, Y_1) \cap L^b(Y_2, Y_1)$, then S^+, S^-, S and $|S|$ are all elements of $L^O(Y_2, Y_1)$.*

An important characterization of $L^N(X_1, X_2)$ can be obtained by combining some of our previous results. We suppose, as always, that $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ form dual systems for the particular choices of Y_1 and Y_2.

Theorem 2.8. *If X_2 is perfect, $Y_1 = X_1^O$ and $Y_2 = X_2^O$, then*

$$L^N(X_1, X_2) = L^b(X_1, X_2) \cap L(X_1, X_2) = L^b(X_1, X_2) \cap L^O(X_1, X_2)$$

Proof. Propositions 2.5, 2.6 and 2.7. Q.E.D.

An interesting corollary to Theorem 2.8 can be obtained by considering the sublattice X^{SO} of X^O consisting of all order sequentially continuous linear functionals.

Corollary 2.8. *In addition to the hypotheses of Theorem 2.8. we suppose that $X_1^O = X_1^{SO}$. Then*

$$L^b(X_1, X_2) \cap L^O(X_1, X_2) = L^b(X_1, X_2) \cap L^{SO}(X_1, X_2).$$

Proof. Since $L^O(X_1, X_2) \subset L^{SO}(X_1, X_2)$, the result will follow from Theorem 2.8 if we show that $L^b(X_1, X_2) \cap L^{SO}(X_1, X_2) \subset L(X_1, X_2)$. Hence, let $T \in L^b(X_1, X_2) \cap L^{SO}(X_1, X_2)$ and let u be any element of Y_2. If $(x_n)_{n=1}^{\infty}$ is a sequence in X_1 which order converges to θ, then $(Tx_n)_{n=1}^{\infty}$ order converges to θ in X_2. Since $Y_2 = X_2^O$, we obtain

$$\lim \langle x_n, T'u\rangle = \lim \langle Tx_n, u\rangle = 0.$$

Therefore, $T'u \in X_1^{SO} \subset Y_1$ which implies that $T \in L(X_1, X_2)$. Q.E.D.

3. *Köthe Function Spaces*

At this point we apply our results to Köthe function spaces. The definitions and notations are those found in MULLINS [5].

Let (E, Σ, μ) be a measure space possessing a cover $\mathfrak{C}$, and let $[\Omega_{\mathfrak{C}}]$ be the vector space of equivalence classes of $\mathfrak{C}$-locally integrable real-valued functions on E (the equivalence classes are obtained by identifying functions which agree almost everywhere on each $C \in \mathfrak{C}$). A partial ordering is defined for elements $[f]$ and $[g]$ in $[\Omega_{\mathfrak{C}}]$ by: $[f] \leqslant [g]$ if $f(x) \leqslant g(x)$ a.e. on each $C \in \mathfrak{C}$. With this partial ordering $[\Omega_{\mathfrak{C}}]$ is always a σ-order complete vector lattice; but, we will suppose that (E, Σ, μ) and $\mathfrak{C}$ are chosen so that $[\Omega_{\mathfrak{C}}]$ is order complete (see [5], Section 2). If Γ is a non-empty subset of $[\Omega_{\mathfrak{C}}]$, its *Köthe dual* Γ^* is defined by

$$\Gamma^* = \{[f] \in [\Omega_{\mathfrak{C}}] : \bar{\int} |f(x)g(x)| d\mu(x) < \infty \text{ for all } [g] \in \Gamma\}$$

(the integral $\bar{\int}$ is obtained from the ordinary integral of (E, Σ, μ) by a process analogous to the one used to obtain the essential integral in BOURBAKI ([2], Chapter 5, § 2), the sets in $\mathfrak{C}$ playing the role of compact sets). A *Köthe function space* is a subset Λ of $[\Omega_{\mathfrak{C}}]$ such that $\Lambda = \Gamma^*$ for some non-empty subset Γ of $[\Omega_{\mathfrak{C}}]$. If Λ is equipped with the partial ordering induced by $[\Omega_{\mathfrak{C}}]$, then Λ is an order complete vector sublattice. We have shown in [5] that Λ^* can be identified with Λ^O for any Köthe space Λ (Theorem 4.8). Moreover, if $\mathfrak{C}$ contains a countable subcover, then $\Lambda^O = \Lambda^{SO}$. Finally, given Köthe spaces Λ_1 and Λ_2, then $L^{SO}(\Lambda_1, \Lambda_2) \subset \subset L^b(\Lambda_1, \Lambda_2)$ ([5] Theorem 5.10). Hence, using the dual systems $\langle \Lambda_1, \Lambda_1^* \rangle$ and $\langle \Lambda_2, \Lambda_2^* \rangle$, we obtain the following consequences to Theorem 2.8 and Corollary 2.8.

Theorem 3.1. *Let (E, Σ, μ) be a measure space and let $\mathfrak{C}$ be a cover for which $[\Omega_{\mathfrak{C}}]$ is order complete. Then, for any Köthe spaces Λ_1 and Λ_2 in $[\Omega_{\mathfrak{C}}]$, we have*

$$L^N(\Lambda_1, \Lambda_2) = L^O(\Lambda_1, \Lambda_2) = L^b(\Lambda_1, \Lambda_2) \cap L(\Lambda_1, \Lambda_2).$$

If, in addition, $\mathfrak{C}$ contains a countable subcover, then

$$L^O(\Lambda_1, \Lambda_2) = L^{SO}(\Lambda_1, \Lambda_2).$$

Note. Several of the results in this paper have been established by FREMLIN [3] for Abstract Köthe Spaces (compare Theorem 4 in [3] with Propositions 2.1 through 2.5).

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